T.D.C. I (Maths)
PART-I

Taylor's Theorem (Finite Form):

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Prestudy: At first, we require to having a general knowledge and study of Limit, continuity and differentiability (Functions of one variable) and theorems related to them (Specially Rolle's theorem).
In this section, we studying General Mean value Theorem, called as Taylor's Theorem (Finite Form) and Maclaurin's theorem with the help of Taylor's theorem.

Statement:- If $f(x)$ is a single valued function of x such that its derivatives upto $f^{n-1}(x)$ are all continuous in $a \le x \le a+h$ and $f^{n}(x)$ exists in $a < x < a+h$ then

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{\lfloor 2}f''(a) + \cdots\cdots + \frac{h^{n-1}}{\lfloor n-1}f^{n-1}(a) + \frac{h^n}{\lfloor n}f^{n}(a+\theta h); \text{ where } 0<\theta<1.$$

Proof: Let us consider the auxiliary function $\phi(x)$ so to

$$\phi(x) = f(x) - f(a) - (x-a)f'(a) - \frac{1}{\lfloor 2}(x-a)^2 f''(a) - \cdots\cdots - \frac{(x-a)^{n-1}}{\lfloor n-1}f^{n-1}(a) + A(x-a)^n \quad \text{——— (i)}$$

Now let us choose A so to $\phi(a+h) = \phi(a)$ ——— (ii)

Putting $x = a+h$ and a in (i); we have

$$\phi(a+h) = f(a+h) - f(a) - (a+h-a)f'(a) - \frac{1}{\lfloor 2}(a+h-a)^2 f''(a) - \cdots\cdots - \frac{(a+h-a)^{n-1}}{\lfloor n-1}f^{n-1}(a) + A(a+h-a)^n$$

$$\phi(a+h) = f(a+h) - f(a) - hf'(a) - \frac{h^2}{\lfloor 2}f''(a) - \cdots - \frac{h^{n-1}}{\lfloor n-1}f^{n-1}(a) + Ah^n \quad \text{---(iii)}$$

and $\phi(a) = f(a) - f(a) - \cdots - \frac{(a-a)^{n-1}}{\lfloor n-1}f^{n-1}(a) + A\cdot(a-a)^n$

$= 0$ ———— (iv)

Using (iii) & (iv) in eqn (ii); we have

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{\lfloor 2}f''(a) + \cdots + \frac{h^{n-1}}{\lfloor n-1}f^{n-1}(a) - Ah^n \quad \text{---(v)}$$

From (ii) & (iv) $\Rightarrow \phi(a+h) = \phi(a) = 0$; $\phi(x)$ satisfies the condition of Rolle's theorem for the interval $(a, a+h)$ and so

$$\phi'(a+\theta_1 h) = 0 \; ; \text{ where } 0 < \theta_1 < 1$$

And also $\phi(a+h) = \phi(a) = \phi'(a) = \phi''(a) = \cdots = \phi^{n-1}(a) = 0$ } —(vi)

(using eqns (i) & (ii))

So, with the help of Rolle's theorem, we have
and continuing above process,

$$\phi^n(x_n) = 0 \; ; \text{where } a < x_n < a+h \quad \text{---(vii)}$$

From eqn (i), we have

$$\phi^n(x_n) = f^n(x_n) + A.\lfloor n$$

using eqn (vii), we have

$$A = -\frac{1}{\lfloor n}f^n(x_n) \quad \text{---(viii)} \qquad \left(\text{As } f^n(x_n) + A.\lfloor n = 0\right)$$

i.e. $A = -\frac{1}{\lfloor n}f^n(a+\theta h)$ —— (ix) (where $x_n = a+\theta h$ & $0 < \theta < 1$)

Using value of A from (ix) into eqn (v), we have

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{\lfloor 2}f''(a) + \cdots + \frac{h^{n-1}}{\lfloor n-1}f^{n-1}(a) + \frac{h^n}{\lfloor n}f^n(a+\theta h)$$

This is required result known as Taylor's theorem and $\frac{h^n}{\lfloor n}f^n(a+\theta h) = R_n(h)$ is known as Lagrange's form of the remainder after the nth terms in the Taylor's expansion of $f(a+h)$.

Maclaurin's Theorem: By Taylor's Theorem, we know that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{\lfloor 2} f''(a) + \cdots + \frac{h^{n-1}}{\lfloor n-1} f^{n-1}(a) + \frac{h^n}{\lfloor n} f^n(a+\theta h) \quad \text{(i)}$$

In eqn (1): putting $a=0$, we have

$$f(h) = f(0) + hf'(0) + \frac{h^2}{\lfloor 2} f''(0) + \cdots + \frac{h^{n-1}}{\lfloor n-1} f^{n-1}(0) + \frac{h^n}{\lfloor n} f^n(\theta h) \quad \text{(ii)}$$

which is statement of Maclaurin's Theorem.

Taylor's Series:

If the derivatives of all orders of $f(x)$ exist in a neighbourhood $(a-\delta, a+\delta)$ of a and $|h|<\delta$ then for all values of n

$$f(a+h) = S_n + R_n \quad \text{(i)}$$

where $S_n = f(a) + hf'(a) + \frac{h^2}{\lfloor 2} f''(a) + \cdots + \frac{h^{n-1}}{\lfloor n-1} f^{n-1}(a)$ — (ii)

and $R_n = \frac{h^n}{\lfloor n} f^n(a+\theta_n h)$ where $0<\theta_n<1$ — (iii)

Now $R_n \to 0$ as $n \to \infty$. Hence we obtained

$$f(a+h) = \lim_{n\to\infty}(S_n) = f(a) + hf'(a) + \frac{h^2}{\lfloor 2} f''(a) + \cdots + \frac{h^n}{\lfloor n} f^n(a) + \cdots \quad \text{(iv)} \quad \left(\text{using (i), (ii) \& (iii)}\right)$$

This is Taylor's series and R_n which tends to 0 is Lagrange's form of the remainder.

Replacing a by x, we have

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{\lfloor 2} f''(x) + \cdots + \frac{h^n}{\lfloor n} f^n(x) + \cdots \quad \text{(v)}$$

Which is Taylor's theorem (for infinite form) or Taylor's expansion series.

Maclaurin's Theorem (For infinite form):—

By Taylor's theorem for infinite form, we know that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{\lfloor 2}f''(a) + \cdots + \frac{h^n}{\lfloor n}f^n(a) + \cdots \quad (1)$$

Putting $a=0$ in (1), we obtained the Maclaurin's theorem

i.e. $$f(h) = f(0) + hf'(0) + \frac{h^2}{\lfloor 2}f''(0) + \cdots + \frac{h^n}{\lfloor n}f^n(0) + \cdots \quad (ii)$$

Here also Maclaurin's form of the remainder is $\frac{h^n}{\lfloor n}f^n(\theta_n h)$ which must tend to zero as n tends to infinity.

Now replacing h by x in (ii); we obtained

$$f(x) = f(0) + xf'(0) + \frac{x^2}{\lfloor 2}f''(0) + \cdots + \frac{x^n}{\lfloor n}f^n(0) + \cdots \quad (iii)$$

which is Maclaurin's series where the Maclaurin's form of the remainder namely $\frac{x^n}{\lfloor n}f^n(\theta_n x)$ tends to zero as n tends to infinity.

Failure of Maclaurin's Theorem: Maclaurin's Theorem fails if

(i) Any one of $f(0), f'(0), \cdots, f^n(0), \cdots$ becomes infinite.

(ii) $f(x)$ or any one of its derivatives, becomes discontinuous for values of x which passes through the value zero.

(iii) the remainder $\frac{x^n}{\lfloor n}f^n(\theta_n x)$ tends to non-zero value as n tends to infinity for in this case the series is divergent.

Failure of Taylor's Theorem: — Taylor's theorem fails if

(i) $f(x)$ or any one of its derivatives, becomes infinite between the values of the given variable (ii) $f(x)$ or any one of its derivatives, becomes the discontinuous for the same values (iii) The remainder $\frac{h^n}{\lfloor n}f^n(a+\theta_n h)$ tends to non zero value as n tends to infinity for in this case the series is divergent.

Example:— Establish the validity of Maclaurin's infinite expansion for e^x.

Solution:— Let $f(x) = e^x$

so $f'(x) = e^x$, $f''(x) = e^x$, ..., $f^n(x) = e^x$, ...

Then by Maclaurin's Theorem, we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{\lfloor 2}f''(0) + \cdots + \frac{x^{n-1}}{\lfloor n-1} f^{n-1}(0) + R_n$$

Where R_n = Maclaurin's form of the remainder

$$= \frac{x^n}{\lfloor n} f^n(\theta_n x) \; ; \quad 0 \nless \theta_n < 1.$$

$$R_n = \frac{x^n}{\lfloor n} e^{\theta_n \cdot x}$$

But We know that $(\lfloor n)^2 > n^n$

$$\Rightarrow \lfloor n > (n^n)^{1/2} = n^{\frac{n}{2}} = (n^{1/2})^n = (\sqrt{n})^n$$

$$\Rightarrow \frac{1}{(\lfloor n)^n} < \frac{1}{(\sqrt{n})^{n^2}} \Rightarrow \frac{1}{\lfloor n} < \frac{1}{(\sqrt{n})^n}.$$

$$\text{So } R_n = \frac{x^n}{\lfloor n} e^{\theta_n \cdot x} < \frac{x^n}{(\sqrt{n})^n} \cdot e^{\theta_n \cdot x}$$

Also if $x > 0$ then $e^{\theta_n \cdot x} < 1 \cdot e^x$ since $0 < \theta_n < 1$

and if $x < 0$, then $e^{\theta_n \cdot x} < 1$ for $0 < \theta_n < 1$

$$\therefore \; |R_n| < \left[\frac{|x|}{\sqrt{n}}\right]^n \cdot e^x \; ; \quad \text{if } x > 0$$

$$|R_n| < \left[\frac{|x|}{\sqrt{n}}\right]^n \quad \text{if } x < 0$$

Thus $|R_n|$ tends to zero as $n \to \infty$ since $\lim_{n\to\infty}\left[\frac{|x|}{\sqrt{n}}\right]^n = 0$

$\Rightarrow |R_n| \to 0$

Hence the expansion of e^x is valid for all values of x.

$\therefore f(x) = e^x = f(0) + x f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \cdots + \frac{x^n}{\lfloor n} f^n(0) + \cdots$

$f(x) = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \cdots + \frac{x^n}{\lfloor n} + \cdots$; since $f^n(0) = 1$ $\forall n \in \mathbb{N}$.

__Exercise:__ Establish the validity of Maclaurin's infinite expansion for $\log(1+x)$.

__Hints:__ $|R_n| = \frac{|x|^n}{n|1+\theta \cdot x|^n} \to 0$; as $n \to \infty$ and $|x| < 1$.

and $f(x) = \log(1+x) = x - \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} \cdots + \frac{(-1)^{n-1}}{n} \cdot x^n + \cdots$